

# Sequential Tests for Exponential Distributions

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*The problem is to test whether the frequency of random events (e.g., DSIF equipment failures) is at a nominally prescribed value. When the actual frequency is higher, a determination of this fact is to be made as quickly as possible. A test based on sequential maximum likelihood ratio methods is developed and approximations of its performance characteristics are derived. Results of Monte Carlo sampling demonstrate that these approximations are accurate and that high statistical efficiency is attained over a broad range of possible higher frequencies. Some applications to reliability and inventory policies for the DSIF are indicated.*

## I. Introduction

If  $X_1, X_2, \dots$  are observed times between random events, then the appropriate model assumes they are independent and identically distributed random variables with a probability density function of the exponential form

$$\lambda e^{-\lambda x}, \quad x > 0$$

The parameter  $\lambda$  is identified as the (average) frequency of events per unit time. In the DSIF, this model is appropriate for both equipment failures and inventory demands. Reliability and inventory policies must depend heavily upon nominal  $\lambda$ -values estimated from available data. Not only are these values subject to error, but also the underlying  $\lambda$ -values themselves are subject to change. Systematic techniques are required for checking whether nominally prescribed  $\lambda$ -values are exceeded. The following examples of specific applications illustrate the general pattern:

- (1) Test inventory demand experience periodically to determine whether minimum stockage levels are sufficient.
- (2) Test demand rates of an individual facility to see if they are in line with the rates in other facilities.
- (3) Check the performance of repair facilities by testing the frequency of repeated failures of repaired parts.
- (4) Compare the reliability performance of an individual supplier's parts against established reliability levels.

The statistical problem arising in such situations is to test the nominal value  $\lambda = \lambda_0$  against a range of alternatives, conveniently written  $\lambda_0(1 + \theta_1) \leq \lambda \leq \lambda_0(1 + \theta_2)$ . Here  $\theta_1$  is the smallest fractional increase in  $\lambda$  worth detecting, and  $\theta_2$  is the largest fractional increase for which high statistical efficiency is important (could be

taken as  $+\infty$ ). Since the problem is essentially unchanged if  $X_1, X_2, \dots$  are multiplied by a scale factor, it will be assumed without loss of generality that  $\lambda_0 = 1$ . When  $\lambda = 1 + \theta$  the probability density function is

$$f_\theta(x) = \begin{cases} (1 + \theta)e^{-(1+\theta)x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

The null and alternative hypotheses are, respectively,

$$H: \theta \geq 0 \text{ and } K: 0 < \theta_1 \leq \theta \leq \theta_2$$

In case  $\theta_1 = \theta_2$ , it is well-known (Ref. 1) that a sequential probability ratio test (SPRT) is optimal in the sense that it minimizes the expected sample sizes required when  $\theta = 0$  and  $\theta = \theta_1$  subject to prescribed type I and type II error probabilities,  $\alpha$  and  $\beta$ . In applications such as reliability testing, however, it frequently is particularly important to reject  $H$  quickly if  $\theta$  is much larger than  $\theta_1$ , so that it is appropriate to choose  $\theta_2$  much larger than  $\theta_1$  and to seek a test which comes reasonably close to minimizing the expected sample sizes for all  $\theta$  in  $[\theta_1, \theta_2]$ . In (Refs. 2 and 3) it is shown that sequential likelihood ratio tests asymptotically minimize all expected sample sizes as  $\alpha, \beta \rightarrow 0$ .

The present paper is concerned with the "small sample" properties of likelihood ratio tests for exponential distributions. Specifically, the tests for  $H$  versus  $K$  above are of the following form. Stop and reject  $H$  at the first  $n$  such that

$$\max_{\theta_1 \leq \theta \leq \theta_2} \log \left( \frac{f_\theta(X_1) \cdots f_\theta(X_n)}{f_0(X_1) \cdots f_0(X_n)} \right) \geq \log \gamma, \quad (\gamma > 1) \quad (1)$$

or equivalently

$$n \log(1 + \hat{\theta}_n) - \hat{\theta}_n S_n \geq \log \gamma$$

where  $S_n = X_1 + \cdots + X_n$  and  $\hat{\theta}_n$  is the value of  $\theta$  which maximizes the likelihood function on  $[\theta_1, \theta_2]$  (Eqs. 3 to 5); stop and accept  $H$  at the first  $n$  such that

$$\log \frac{f_{\theta_1}(X_1) \cdots f_{\theta_1}(X_n)}{f_0(X_1) \cdots f_0(X_n)} \leq \log \xi < 0$$

that is,

$$n \log(1 + \theta_1) - \theta_1 S_n \leq \log \xi \quad (2)$$

The main problem is to determine how to choose  $\gamma$  and  $\xi$  to attain prescribed bounds,  $\alpha$  and  $\beta$ , on type I and type II error probabilities. The principal results in this connection are Eqs. (7) and (25) and Table 2 of Section 3. Several examples were investigated by Monte Carlo methods and the formulas proved to be acceptably accurate in all cases. The results are given in Table 3 of *Subsection III* and examples indicating lower bounds on the efficiency of the likelihood ratio tests are given in Table 4.

A major part of the derivation of the approximate formulas for error probabilities is the approximation in *Subsection II* of the distribution of the "excess over the boundary" or "overshoot" when Eqs. (1) and (2) are first satisfied. In the standard approximations to the error probabilities and expected sample sizes of SPRT's, the effects of this overshoot are ignored; but any reasonably accurate approximation must take it into account, as the results of Ref. 4 indicate. It is well-known that the excess over the boundary in Eq. (2) is exponentially distributed by virtue of the characteristic "memoryless" property of the exponentially distributed  $X$ 's. The derivation of this result is given in *Subsection II*. The distribution of excess over the boundary in Eq. (1) when  $\hat{\theta}_n \equiv \theta_1$  (i.e.,  $\theta_1 = \theta_2$ ) is approximated by deriving the limit distribution as  $\log \gamma \rightarrow \infty$ , based on a result of S. C. Port (Ref. 5). This limit distribution also leads to a natural approximation to the excess in Eq. (1) in the general case.

A major simplification is effected in *Subsection III* by studying the probability when  $\theta = 0$  that Eq. (1) *ever holds* for any  $n$ . This differs from the probability,  $\alpha$ , that Eq. (1) occurs before Eq. (2) by *at most* a factor of  $(1 - \beta)$ , by virtue of the argument in the usual derivation of SPRT error approximations (Ref. 6). In typical applications this factor is on the order of 0.95 and may be neglected without serious harm. The approximations to SPRT error probabilities given in Section 2 take into account this small correction, but no attempt is made to do this for likelihood ratio tests because the effect is much more complicated than a simple factor and is difficult to determine.

A convenient description of the likelihood ratio defined by Eqs. (1) and (2) is the following: Perform the SPRT of  $\theta = 0$  versus  $\theta = \theta_1$  defined by the inequalities

$$\log \xi < n \log(1 + \theta_1) - \theta_1 S_n < \log \gamma, \quad (3)$$

stopping to make the appropriate decision as soon as either inequality is violated. In addition, whenever

$S_n/n < (1 + \theta_2)^{-1}$  (so that  $\hat{\theta}_n = \theta_2$ ) reject  $H$  if

$$n \log(1 + \theta_2) - \theta_2 S_n \geq \log \gamma \quad (4)$$

and whenever

$$(1 + \theta_2)^{-1} \leq \frac{S_n}{n} < (1 + \theta_1)^{-1}$$

so that  $1 + \hat{\theta}_n = n/S_n$ , reject  $H$  if

$$\frac{S_n}{n} - \log\left(\frac{S_n}{n}\right) \geq 1 + \frac{\log \gamma}{n} \quad (5)$$

(Note that Eq. (5) can hold when  $S_n/n > 1$ , but  $H$  is not rejected in this case.) The likelihood ratio test thus modifies the SPRT of  $\theta = 0$  versus  $\theta = \theta_1$  by rejecting  $\theta = 0$  sooner when  $\hat{\theta}_n > \theta_1$ . For

$$n > \frac{\log \gamma}{\log(1 + \theta_1) - \frac{\theta_1}{1 + \theta_1}}$$

(the approximate expected sample size of the SPRT when  $\theta = \theta_1$ ) it is routinely verified that neither Eqs. (4) nor (5) can hold unless the right-hand inequality of Eq. (3) is violated. Thus for  $n$  this large, the statistician goes back to performing an SPRT. Obviously, the usual discussions of truncation apply (e.g., Ref. 6), and the effect of truncation on error probabilities is no greater than for the SPRT. Other uses of likelihood ratio tests, such as for testing composite  $H$  or minimizing expected sample sizes over the region between  $H$  and  $K$ , are discussed in Ref. 2. The results of *Subsection III* can be applied in these cases also.

In the limiting case as  $\beta \rightarrow 0$ , one has what Herbert Robbins has called an "open-ended test" of  $H$ : observations are terminated only if Eq. (1) is satisfied (and  $H$  is rejected). The type I error probability,  $\alpha$ , now represents the probability of ever stopping when  $\theta = 0$  (which is approximated in *Subsection III*). Such a test (or a test with very small  $\beta$ ) may be called for if observations are inexpensive or mandatory (perhaps for another use), and if, for example, *one is already acting as if the null hypothesis were true*. Examples of this situation are the monitoring of repeated failures of repairable equipment to determine eventual replacement, and the continuous evaluation of failure data or test data on a product to determine whether its acceptance or use should be discontinued (or subjected to more stringent evaluation).

## II. Distribution of Excess Over the Boundary

In this section the case  $\theta_1 = \theta_2$  (i.e., an SPRT) is considered throughout. Let  $N$  be the number of observations required to terminate and consider first the excess over the boundary when (2) occurs; that is, the quantity

$$\begin{aligned} & \log \xi - N \log(1 + \theta_1) + \theta_1 S_N \\ &= [\log \xi - N \log(1 + \theta_1) + \theta_1 S_{N-1}] + \theta_1 X_N > 0 \end{aligned}$$

Now, conditioning on the value, say  $r$ , of the bracketed quantity (which must be negative, or else stopping would have occurred before  $N$ ), and on the event  $N = n$ , the distribution of the excess (if  $\theta = 0$ ) is that of  $r + \theta_1 X_n$  given the latter is positive ( $X_n$  being independent of  $X_1, \dots, X_{n-1}$ ). By the "memoryless" property of the exponential distribution, the conditional distribution of  $X_n + r/\theta_1$  given it is positive is exponential, mean 1, for every  $r$  and  $n$ ; hence, when  $\theta = 0$  the distribution of the excess is exponential, mean  $\theta_1$  (i.e., that of  $\theta_1 X_1$ ).

By Wald's derivation of SPRT error approximations (Ref. 6)

$$\frac{P_{\theta_1}(\text{decide } \theta = 0)}{P_0(\text{decide } \theta = 0)} = \frac{\beta}{1 - \alpha} = E_0 \left[ \frac{f_{\theta_1 N}}{f_{0N}} \mid \text{decide } \theta = 0 \right] \quad (6)$$

where  $f_{\theta_1 N}/f_{0N}$  is the value of the likelihood ratio upon stopping. Let

$$L_n = n \log(1 + \theta_1) - \theta_1 S_n$$

Then Eq. (6) can be written

$$\begin{aligned} \frac{\beta}{1 - \alpha} &= E_0[\exp(L_N) \mid \text{decide } \theta = 0] \\ &= \xi E_0[\exp(-(\log \xi - L_N)) \mid \text{decide } \theta = 0] \\ &= \xi E_0 \exp(-\theta_1 X_1) \end{aligned}$$

since the excess over the boundary,  $\log \xi - L_N$ , is distributed like  $\theta_1 X_1$ . By routine calculation

$$\frac{\beta}{1 - \alpha} = \frac{\xi}{1 + \theta_1} \quad (7)$$

The analog of Eq. (6) is

$$\frac{P_0(\text{decide } \theta = \theta_1)}{P_{\theta_1}(\text{decide } \theta = \theta_1)} = \frac{\alpha}{1 - \beta} = E_{\theta_1} \left[ \frac{f_{0N}}{f_{\theta_1 N}} \mid \text{decide } \theta = \theta_1 \right] \quad (8)$$

and it follows as above that

$$\frac{\alpha}{1-\beta} = \gamma^{-1} E_{\theta_1} [\exp(- (L_N - \log \gamma)) | \text{decide } \theta = \theta_1] \quad (9)$$

The approximate formula (11) for  $\alpha/(1-\beta)$  will be derived from Eq. (9) by replacing the excess over the boundary,  $L_N - \log \gamma$ , by  $R$ , a random variable whose distribution is the limit distribution of  $L_N - \log \gamma$  as  $\log \gamma \rightarrow \infty$ . A routine argument shows that this limit distribution does not depend on  $\log \xi$ , being always the

same as in the case  $\log \xi = -\infty$ , i.e., the usual one-sided boundary in renewal theory.

By Theorem 3 of Ref. 5 the limit distribution of excess over the boundary for cumulative sums of non-lattice variables (e.g., the sequence  $\{L_n\}$ ) has density  $e(-x)/E_{\theta_1} L_1$ , where

$$e(-x) = P_{\theta_1}(L_1, L_2, \dots > x) \quad \text{for } x \geq 0$$

and is zero for  $x < 0$ . The density function of  $L_1 = \log(1 + \theta_1) - \theta_1 X_1$  when  $\theta = \theta_1$  is

$$g(t) = \begin{cases} \frac{1 + \theta_1}{\theta_1} \exp\left(\frac{1 + \theta}{\theta} [t - \log(1 + \theta)]\right), & t \leq \log(1 + \theta) \\ 0, & t > \log(1 + \theta) \end{cases}$$

Conditioning on the value of  $L_1$ , we have

$$e(-x) = \int_x^\infty \left(1 - \frac{e^{x-t}}{1 + \theta_1}\right) g(t) dt$$

since

$$P_{\theta_1}[L_2, L_3, \dots > x | L_1 = t > x] = P_{\theta_1}[L_1, L_2, \dots > x - t]$$

which is just a limiting case of Eq. (7) with  $\log \gamma = +\infty$  (so that  $\alpha = 0$ ) and  $x - t = \log \xi$ . The integral is easily evaluated, yielding

$$e(-x) = \begin{cases} 1 - \frac{e^x}{1 + \theta_1}, & 0 \leq x \leq \log(1 + \theta_1) \\ 0, & \text{otherwise} \end{cases}$$

Since

$$E_{\theta_1} L_1 = \log(1 + \theta_1) - \frac{\theta_1}{(1 + \theta_1)}$$

the limiting distribution of the excess over the boundary,  $L_N - \log \gamma$ , has density

$$h(x) = \begin{cases} \frac{1 + \theta_1 - e^x}{(1 + \theta_1) \log(1 + \theta_1) - \theta_1}, & 0 \leq x \leq \log(1 + \theta_1) \\ 0, & \text{otherwise} \end{cases} \quad (10)$$

Therefore, the approximation derived from Eq. (9) is

$$\frac{\alpha}{1-\beta} \approx \gamma^{-1} E_{\theta_1} \exp(-R) = \gamma^{-1} \frac{\theta_1 - \log(1 + \theta_1)}{(1 + \theta_1) \log(1 + \theta_1) - \theta_1} \quad (11)$$

where  $R$  has density  $h$ .

Combining Eqs. (7) and (11) leads to the approximations

$$\alpha \approx \frac{1 + \theta_1 - \xi}{\gamma G(\theta_1) (1 + \theta_1) - \xi} \quad (12)$$

and

$$1 - \beta \approx \gamma G(\theta_1) \alpha \approx \frac{\gamma G(\theta_1) (1 + \theta_1 - \xi)}{\gamma G(\theta_1) (1 + \theta_1) - \xi} \quad (13)$$

where

$$G(\theta_1) = \frac{(1 + \theta_1) \log(1 + \theta_1) - \theta_1}{\theta_1 - \log(1 + \theta_1)}$$

Relations (7) and (11) also yield approximations for the expected sample sizes, based on Wald's equation,  $E L_1 \cdot E N = E L_N$ .

$$\begin{aligned} \left( \log(1 + \theta_1) - \frac{\theta_1}{1 + \theta_1} \right) E_{\theta_1} N &\approx \beta (\log \xi - E_{\theta_1}(\theta_1 X_1)) + (1 - \beta) (\log \gamma + E_{\theta_1} R) \\ &= \beta \left( \log \xi - \frac{\theta_1}{1 + \theta_1} \right) + (1 - \beta) \left( \log \gamma + \frac{\frac{1}{2}(1 + \theta_1)[\log(1 + \theta_1)]^2}{(1 + \theta_1) \log(1 + \theta_1) - \theta_1} - 1 \right) \end{aligned} \quad (14)$$

Similarly,

$$\begin{aligned} (\log(1 + \theta_1) - \theta_1) E_0 N &\approx (1 - \alpha) (\log \xi - E_0(\theta_1 X_1)) \\ &\quad + \alpha (\log \gamma + E_0 R) \end{aligned}$$

The value of  $E_0 R$  does not come out of the above derivation. However, its effect is small, since it is multiplied by  $\alpha$  and is in any case bounded by zero and  $\log(1 + \theta_1)$ . The latter value gives a smaller approximation (since  $\log(1 + \theta_1) - \theta_1$  is negative):

$$\begin{aligned} (\log(1 + \theta_1) - \theta_1) E_0 N &\approx (1 - \alpha) (\log \xi - \theta_1) \\ &\quad + \alpha (\log \gamma + \log(1 + \theta_1)) \end{aligned} \quad (15)$$

The approximations (12) to (15) are useful in *Subsection III* for deriving similar approximations in the case of likelihood ratio tests. For SPRTs, Refs. 4 and 7 give exact error probabilities and expected sample sizes. However, formulas (11) to (13) together with the values of  $G(\theta)^{-1}$  in Table 2 of *Subsection III* are quite useful in obtaining reasonably accurate approximations to the values of  $\gamma$  and  $\xi$  needed to get prescribed  $\alpha$  and  $\beta$  for SPRTs. The following table gives some examples to illustrate the degree of accuracy of the approximations (11) to (15).

### III. Approximate Error Probabilities of Likelihood Ratio Tests

The type II error probabilities of the likelihood ratio tests defined by Eqs. (1) and (2) obviously attain a maximum,  $\beta$ , at  $\theta = \theta_1$ . To approximate  $\beta$ , note that the derivation of Eqs. (6) and (7) applies, so that Eq. (7) holds with  $\alpha$  equal to the type I error probability of the likelihood ratio test. The value of the factor  $(1 - \alpha)$  in Eq. (7) can be approximated using the principal result (25) of this section, but the effect on the determination of  $\beta$  is small in typical problems, so that  $\beta \approx \xi/(1 + \theta_1)$ .

To approximate  $\alpha$ , an approximation to

$$\alpha^* = P_0 \left( \max_{\theta_1 \leq \theta \leq \theta_2} [n \log(1 + \theta) - \theta S_n] \geq \log \gamma \text{ for some } n \right)$$

will suffice, since the ratio  $\alpha/\alpha^*$  is at least  $1 - \beta$ , as discussed in *Subsection I*. The inequality

$$\max_{\theta_1 \leq \theta \leq \theta_2} [n \log(1 + \theta) - \theta S_n] \geq \log \gamma$$

is clearly equivalent to

$$S_n \leq \max_{\theta_1 \leq \theta \leq \theta_2} \left[ \frac{n \log(1 + \theta)}{\theta} - \frac{\log \gamma}{\theta} \right] \quad (16)$$

and differentiation of the bracketed quantity in Eq. (16) shows that the maximum is attained at  $\tilde{\theta}_n$ , the solution of

$$\frac{\log \gamma}{n} = \log(1 + \tilde{\theta}_n) - \frac{\tilde{\theta}_n}{1 + \tilde{\theta}_n} = I(\tilde{\theta}_n), \text{ say}$$

if  $m \leq n \leq M$ , where  $m$  is the largest integer  $\leq (\log \gamma)/I(\theta_2)$  and  $M$  is the smallest integer  $\geq (\log \gamma)/I(\theta_1)$ . If  $n < m$  the maximum is attained at  $\theta_2$  and if  $n > M$  it is attained at  $\theta_1$ . Let  $N$  be the smallest  $n$  for which Eq. (16) is satisfied (or  $\infty$  if there is no  $n$ ) and let  $N(\theta)$  for  $\theta > 0$  be the smallest  $n$  (or  $\infty$  if there is no  $n$ ) such that

$$S_n \leq \frac{n \log(1 + \theta)}{\theta} - \frac{\log \gamma}{\theta} \quad (17)$$

If  $N = n$ , then since  $\tilde{\theta}_n$  maximizes the bracketed quantity in Eq. (16), evidently Eq. (17) holds with  $\theta = \tilde{\theta}_n$ , so that  $N(\tilde{\theta}_n) \leq n$  and, in fact,  $N(\tilde{\theta}_n) = n$  ( $N$  being  $\leq N(\tilde{\theta}_n)$  by virtue of  $\tilde{\theta}_n \in [\theta_1, \theta_2]$ ). Therefore, for all  $n$

$$P_0(N = n) \leq P_0(N(\tilde{\theta}_n) = n) \quad (18)$$

If  $m = 0$ , then  $\theta_2 > \theta_1$  and the values of  $\theta$  in  $[\tilde{\theta}_1, \theta_2]$  play no role in the test since the maximum in Eq. (16) is the same for every  $n$  if  $\theta_2$  is replaced by  $\tilde{\theta}_1$ . Thus, there is no loss of generality in assuming that  $\theta_2 \leq \tilde{\theta}_1$ , i.e.,

that  $m \geq 1$ . Since  $\tilde{\theta}_1 = \dots = \tilde{\theta}_m = \theta_2$  and  $\tilde{\theta}_M = \tilde{\theta}_{M+1} = \dots = \theta_1$ , it follows from Eq. (18) that

$$\begin{aligned} \alpha^* &= \sum_{n=1}^{\infty} P_0(N = n) \leq P_0(N(\theta_2) \leq m) \\ &\quad + P_0(M \leq N(\theta_2) < \infty) \\ &\quad + \sum_{n=m+1}^{M-1} P_0(N(\tilde{\theta}_n) = n) \end{aligned} \quad (19)$$

Regarding  $n$  as a continuous variable  $u$  on the interval from  $(\log \gamma)/I(\theta_2) \approx m$  to  $(\log \gamma)/I(\theta_1) \approx M$ , with  $I(\tilde{\theta}_u) \equiv (\log \gamma)/u$ , one can approximate the last summation in Eq. (19) by an integral, which yields the approximate upper bound

$$\begin{aligned} \alpha^* &\leq P_0\left(N(\theta_2) \leq \frac{\log \gamma}{I(\theta_2)}\right) + P_0\left(\frac{\log \gamma}{I(\theta_1)} \leq N(\theta_1) < \infty\right) \\ &\quad + \int_{\log \gamma / I(\theta_2)}^{\log \gamma / I(\theta_1)} P_0(N(\tilde{\theta}_u) = u) du \end{aligned} \quad (20)$$

(The event  $N(\tilde{\theta}_u) = u$  makes sense only if  $u$  is an integer, but a natural interpolation will come from the calculations which follow). A relation similar to Wald's relation (8), derivable by a similar "cancellation of densities" argument (Ref. 6), is the following. For integer  $u$ ,

$$\frac{P_0(N(\tilde{\theta}_u) = u)}{P_{\tilde{\theta}_u}(N(\tilde{\theta}_u) = u)} = E_{\tilde{\theta}_u}[\exp(-L_u) | N(\tilde{\theta}_u) = u] \quad (21)$$

where  $L_u = u \log(1 + \tilde{\theta}_u) - \tilde{\theta}_u S_u$ .

The random variable  $N(\tilde{\theta}_u)$  is by definition the number of observations of a one-sided SPRT of  $\theta = 0$  versus  $\tilde{\theta} = \theta_u$ . This suggests the approximation

$$\begin{aligned} E_{\tilde{\theta}_u}[\exp(-L_u) | N(\tilde{\theta}_u) = u] &\approx E_{\tilde{\theta}_u}[\exp(-\log \gamma - R(\tilde{\theta}_u))] \\ &= \gamma^{-1} E_{\tilde{\theta}_u} \exp(-R(\tilde{\theta}_u)), \end{aligned}$$

where  $R(\tilde{\theta}_u)$  has the limit distribution of excess over the boundary derived in *Subsection II* (upon setting  $\theta_1 = \tilde{\theta}_u$ ). Applying this approximation and Eq. (21),

$$\frac{P_0(N(\tilde{\theta}_u) = u)}{P_{\tilde{\theta}_u}(N(\tilde{\theta}_u) = u)} \approx \gamma^{-1} \frac{\tilde{\theta}_u - \log(1 + \tilde{\theta}_u)}{(1 + \tilde{\theta}_u) \log(1 + \tilde{\theta}_u) - \tilde{\theta}_u} \quad (22)$$

An approximation to  $P_{\tilde{\theta}_u}(N(\tilde{\theta}_u) = u)$  is suggested by the fact (Ref. 6) that when  $\tilde{\theta}_u$  is true and  $\gamma$  is large,  $N(\tilde{\theta}_u)$  is

approximately normally distributed with mean  $(\log \gamma)/I(\tilde{\theta}_u) = u$  and variance

$$\frac{(\log \gamma) \text{Var}_{\tilde{\theta}_u} \left( \log \left( \frac{f_{\tilde{\theta}_u}(X)}{f_0(X)} \right) \right)}{\left( E_{\tilde{\theta}_u} \log \left( \frac{f_{\tilde{\theta}_u}(X)}{f_0(X)} \right) \right)^3} = \frac{(\log \gamma) \frac{\tilde{\theta}_u^2}{(1 + \tilde{\theta}_u)^2}}{I(\tilde{\theta}_u)^3}$$

Taking  $1/\sqrt{2\pi}$  times the reciprocal of the standard deviation as an approximation to the probability of the unit-length interval centered on  $u$ ,

$$\begin{aligned} P_{\tilde{\theta}_u}(N(\tilde{\theta}_u) = u) &\approx \frac{I(\theta_u)^{3/2} (1 + \tilde{\theta}_u)}{\sqrt{2\pi} (\log \gamma)^{1/2} \tilde{\theta}_u} \\ &= \frac{I(\tilde{\theta}_u)}{\sqrt{2\pi} u} \cdot \frac{1 + \tilde{\theta}_u}{\tilde{\theta}_u} \end{aligned}$$

since

$$\frac{I(\tilde{\theta}_u)^{1/2}}{(\log \gamma)^{1/2}} = u^{-1/2}$$

Combining with Eq. (22) yields

$$\begin{aligned} P_0(N(\tilde{\theta}_u) = u) &\approx \frac{\gamma^{-1} I(\tilde{\theta}_u)}{\sqrt{2\pi} u} \cdot \frac{1 - \frac{(\log(1 + \tilde{\theta}_u))}{\tilde{\theta}_u}}{\log(1 + \tilde{\theta}_u) - \frac{\tilde{\theta}_u}{(1 + \tilde{\theta}_u)}} \\ &= \frac{\gamma^{-1}}{\sqrt{2\pi} u} \left( 1 - \frac{\log(1 + \tilde{\theta}_u)}{\tilde{\theta}_u} \right). \end{aligned} \quad (23)$$

Similarly, the distribution of  $N(\theta_2)$  when  $\theta = \theta_2$  is asymptotically normal with mean  $\log \gamma / I(\theta_2)$ , which suggests

$$P_0(N(\theta_2) \leq (\log \gamma)/I(\theta_2)) \approx \frac{1}{2} \gamma^{-1} \cdot \frac{\theta_2 - \log(1 + \theta_2)}{(1 + \theta_2) \log(1 + \theta_2) - \theta_2}$$

Using a similar approximation for  $P_0((\log \gamma)/I(\theta_1) \leq N(\theta_1) < \infty)$  and plugging Eq. (23) into Eq. (20),

$$\begin{aligned} \alpha^* &\leq \frac{1}{2} \gamma^{-1} \left[ \frac{\theta_2 - \log(1 + \theta_2)}{(1 + \theta_2) \log(1 + \theta_2) - \theta_2} \right. \\ &\quad \left. + \frac{\theta_1 - \log(1 + \theta_1)}{(1 + \theta_1) \log(1 + \theta_1) - \theta_1} \right] \\ &\quad + \frac{\gamma^{-1}}{\sqrt{2\pi}} \int_{(\log \gamma)/I(\theta_2)}^{(\log \gamma)/I(\theta_1)} u^{-1/2} \left( 1 - \frac{\log(1 + \tilde{\theta}_u)}{\tilde{\theta}_u} \right) du \end{aligned} \quad (24)$$

The change of variable  $x = \log(1 + \tilde{\theta}_u)$  leads to

$$\alpha^* \leq \frac{1}{2} \gamma^{-1} \left[ \frac{\theta_2 - \log(1 + \theta_2)}{(1 + \theta_2) \log(1 + \theta_2) - \theta_2} + \frac{\theta_1 - \log(1 + \theta_1)}{(1 + \theta_1) \log(1 + \theta_1) - \theta_1} \right] + \frac{\gamma^{-1} \sqrt{\log \gamma}}{\sqrt{2\pi}} \int_{\log(1 + \theta_1)}^{\log(1 + \theta_2)} \frac{e^{-x}(e^x - 1 - x)}{(x + e^{-x} - 1)^{3/2}} dx \quad (25)$$

Let

$$Q(\theta) = \int_{\log(1, 0.1)}^{\log(1 + \theta)} \frac{e^{-x}(e^x - 1 - x)}{(x + e^{-x} - 1)^{3/2}} dx$$

Given  $\gamma, \theta_1, \theta_2$ , the approximate upper bound on  $\alpha^*$ , Eq. (25), and hence the approximate upper bound on  $\alpha$  for the likelihood ratio test determined by  $\gamma, \theta_1, \theta_2$ , can be written

$$\frac{1}{2} \gamma^{-1} (G(\theta_1)^{-1} + G(\theta_2)^{-1}) + \frac{\gamma^{-1} \sqrt{\log \gamma}}{\sqrt{2\pi}} (Q(\theta_2) - Q(\theta_1)) \quad (26)$$

Monte Carlo sampling was performed in five cases to compare the approximate upper bound in Eq. (25) with the observed frequency of type I errors. The results are given in Table 3 with the tolerances in the last column being the standard deviations of the observed frequencies. All probabilities are expressed as percentages.

For obvious reasons, the Monte Carlo experiments could not be used to determine the frequency with which Eq. (16) holds for some  $n = 1, 2, \dots$ . The experiments were carried out for a likelihood ratio test which could accept the null hypothesis (and thereby terminate the sample sequence) as soon as Eq. (2) was satisfied. Values of  $\xi$  were chosen to ensure that the effect of these terminations was small compared to the sampling errors. In all but the second of the five cases in Table 3, the observed frequencies were smaller than the approximate upper bounds, and the agreement between the two would be improved if the sample sequences were not terminated.

An analysis of the efficiencies of the likelihood ratio tests is easy to make using a definition slightly different

from the usual definition. The one-sided SPRT of 0 versus  $\theta$  defined by Eq. (17) has the optimality property of minimizing  $E_\theta N$  among all tests with  $\alpha \leq \tilde{\alpha}$ , its type I error probability. The efficiency of a test with  $\alpha = \tilde{\alpha}$  is usually defined as the ratio  $E_\theta N / E_\theta N' < 1$ , where  $N'$  is the number of observations taken by the other test. But Wald's approximation for expected sample sizes (Ref. 6) and Eq. (14) both indicate that  $E_\theta N$  is very nearly proportional to  $\log \tilde{\alpha}^{-1}$ . It is therefore reasonable (and, in the present investigation, convenient) to define the efficiency of a test with given  $\alpha$  and  $E_\theta N'$  as the ratio  $\log \alpha^{-1} / \log \tilde{\alpha}^{-1}$  where  $\tilde{\alpha}$  is the type I error probability of a one-sided SPRT of 0 versus  $\theta$  having  $E_\theta N = E_\theta N'$ .

For the likelihood ratio tests it is of interest to compute these efficiencies for  $\theta$ 's ranging over  $[\theta_1, \theta_2]$ . This is an "unfair" comparison in the sense that a single likelihood ratio test, designed to perform well for all  $\theta$  in  $[\theta_1, \theta_2]$  is compared for each  $\theta$  with the optimal test (an SPRT) *chosen especially for that  $\theta$* . Nevertheless, the comparison is interesting as an indication of how great a price is paid in loss of efficiency at one value of  $\theta$  in order to attain simultaneously "good" performance over a broad range of  $\theta$ 's.

Let  $\alpha$  denote the type I error probability of a given likelihood ratio test. When  $\theta_1$  is true, the expected number of observations is by Eq. (1) no larger than that of a one-sided SPRT of 0 versus  $\theta_1$  with the same boundary,  $\log \gamma$ . Hence, a lower bound on the efficiency (as defined above) is

$$\frac{\log \alpha^{-1}}{\log \alpha(\theta_1)^{-1}}$$

where  $\alpha(\theta_1)$  is the type I error probability of the SPRT of 0 versus  $\theta_1$  with boundary  $\log \gamma$ . Similarly, define for  $\theta$  in  $[\theta_1, \theta_2]$

$$e^*(\theta) = \frac{\log \alpha^{-1}}{\log \alpha(\theta)^{-1}}$$

where  $\alpha(\theta)$  is the type I error probability of the one-sided SPRT of 0 versus  $\theta$  with boundary  $\log \gamma$ . Table 4 of  $e^*(\theta)$ 's was computed by using the approximate upper bound (25) as  $\alpha$  and approximating  $\alpha(\theta)$  from Eq. (13) as  $\gamma^{-1}/G(\theta)$  ( $\beta = 0$ ). Since  $e^*(\theta)$  was found to be nearly constant over  $[\theta_1, \theta_2]$ , decreasing slightly over the interval, only the values of  $e^*(\theta_1)$  and  $e^*(\theta_2)$  are given.

#### IV. Conclusion

From Table 2 it is easy to determine the critical values  $\gamma$  and  $\xi$  needed to achieve prescribed error probabilities with a sequential likelihood ratio test. As the results in

Table 4 indicate, a test determined in this way will guarantee high statistical efficiency even if the range  $[\theta_1, \theta_2]$  is broad. The application of these tests for DSIF reliability and inventory policies along the lines in *Subsection I* should be both practical and useful.

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**Table 1. Comparison of operating characteristics of SPRTs with approximations (11) to (15)**

$\theta$	$\log \gamma$	$\xi$	$\alpha^*, \%$	$\beta, \%$	$E_0 N$	$E_1 N$
1.0	2.48	0.005	6.6 (6.6) <sup>a</sup>	0.23 (0.23)	18.5 (18.6)	14.0 (14.0)
0.7	3.73	0.02	2.0 (2.0)	1.2 (1.2)	26.2 (26.2)	32.1 (32.1)
0.4	3.39	0.10	2.8 (2.8)	6.9 (6.9)	39.7 (39.8)	60.7 (60.9)
0.3	2.48	0.10	7.1 (7.0)	7.1 (7.2)	59.0 (59.4)	69.8 (69.8)

<sup>a</sup>Actual values (in parentheses) were computed using the exact formulas in Ref. 7. The agreement is very good and gets better as  $\log \gamma$  increases because the limit distribution of excess over the boundary is quickly approached.

**Table 3. Comparison of error probability approximations with Monte Carlo experiments**

$\theta_1$	$\theta_2$	$\log \gamma$	Approximation, %	Observed frequency, %
0.3	1	2.48	14.0	$13.2 \pm 1.1$
0.2	3	3.39	9.01	$9.09 \pm 0.91$
0.5	4	3.70	4.99	$4.33 \pm 0.46$
0.7	2	3.73	3.50	$3.34 \pm 0.28$
0.3	1	5.31	1.00	$0.83 \pm 0.14$

**Table 4. Approximate lower bounds on efficiency, %**

$\theta_1$	$\theta_2$	$\log \gamma$	$e^*(\theta_1)$	$e^*(\theta_2)$
0.3	1	2.48	77	73
0.2	3	3.39	70	63
0.5	4	3.70	78	71
0.7	2	3.73	86	82
0.3	1	5.31	85	83

**Table 2. Values of  $G(\theta) \frac{\theta - \log(1 + \theta)}{(1 + \theta)\log(1 + \theta) - \theta}$**

$$\text{and } Q(\theta) = \int_{\log(1.01)}^{\log(1+\theta)} \frac{e^{-x}(e^x - 1 - x)}{(x + e^{-x} - 1)^{3/2}} dx$$

$\theta$	$G(\theta_1)^{-1}$	$Q(\theta)$	$\theta$	$G(\theta)^{-1}$	$Q(\theta)$	$\theta$	$G(\theta)^{-1}$	$Q(\theta)$
0.02	0.993	0.971	0.26	0.926	4.396	0.70	0.838	5.501
0.03	0.990	1.535	0.27	0.923	4.442	0.72	0.835	5.529
0.04	0.987	1.933	0.28	0.921	4.486	0.74	0.832	5.556
0.05	0.984	2.239	0.29	0.919	4.528	0.76	0.829	5.583
0.06	0.981	2.488	0.30	0.916	4.568	0.78	0.826	5.608
0.07	0.978	2.697	0.32	0.912	4.644	0.80	0.822	5.632
0.08	0.975	2.877	0.34	0.907	4.715	0.85	0.815	5.690
0.09	0.972	3.035	0.36	0.903	4.782	0.90	0.808	5.744
0.10	0.969	3.175	0.38	0.898	4.844	0.95	0.801	5.794
0.11	0.966	3.301	0.40	0.894	4.902	1.00	0.794	5.841
0.12	0.963	3.416	0.42	0.890	4.958	1.1	0.782	5.926
0.13	0.960	3.521	0.44	0.886	5.010	1.2	0.770	6.001
0.14	0.957	3.617	0.46	0.882	5.059	1.3	0.759	6.068
0.15	0.954	3.706	0.48	0.878	5.106	1.4	0.748	6.129
0.16	0.952	3.789	0.50	0.874	5.150	1.5	0.738	6.184
0.17	0.949	3.867	0.52	0.870	5.192	1.6	0.729	6.234
0.18	0.946	3.939	0.54	0.866	5.233	1.8	0.711	6.323
0.19	0.944	4.008	0.56	0.862	5.271	2.0	0.696	6.399
0.20	0.941	4.072	0.58	0.859	5.308	2.5	0.662	6.549
0.21	0.938	4.133	0.60	0.855	5.344	3.0	0.634	6.662
0.22	0.936	4.191	0.62	0.852	5.378	3.5	0.611	6.751
0.23	0.933	4.246	0.64	0.848	5.410	4.0	0.591	6.823
0.24	0.931	4.298	0.66	0.845	5.442	4.5	0.573	6.884
0.25	0.928	4.348	0.68	0.841	5.472	5.0	0.558	6.935